

The Index Theorem

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Mathematics is both an Art and a Science and beauty plays an essential role, a fact recognized by all mathematicians. The great German mathematician Hermann Weyl, one of my heroes, put it well when he said

“My work has always tried to unite the true with the beautiful and when I had to choose one or the other I usually chose the beautiful.”

Since mathematics is the most precise of the sciences and is devoted to finding out the truth, Weyl’s statement might appear bizarre and even provocative – a tongue-in-cheek remark. But I believe Weyl was quite serious. The apparent paradox in Weyl’s dictum is that objective truth is what we all search for but, at any stage, it is uncertain and provisional. But beauty, which is a subjective experience “in the eye of the beholder,” is the light we follow in the hope that it is leading us to truth.

But what is beauty in mathematics, and is it similar to beauty in art, music or poetry? Karl Weierstrass, who was outwardly an austere analyst once said that

“It is not possible to be a complete mathematician without having the soul of a poet.”

Such statements are hard for outsiders to understand, though I have argued that the famous equation of Euler,

$$e^{2\pi i} = 1$$

is in its brevity and depth, the equivalent of Hamlet’s famous question,

“To be or not to be...”

But perhaps architecture is the art most comparable to mathematics, where there is a grand vision full of delicate detail, where solid foundations and functional utility are all essential components.

The equation I have chosen to epitomize beauty in my own work has the grandeur that comes from a rich history and multiple connections to different branches of mathematics: topology, geometry, analysis. But the delicate details of the arguments, whose simplicity deceptively covers hidden depths, is only apparent to the craftsmen who can appreciate them.

Like a building with three towers the equation has three terms, from different parts of mathematics, linked together in a striking manner. As with great architecture it has features that trace their roots far back in time, while at the same time embody the latest techniques and point towards the future.

The ancestry of the equation connects it to many royal families: to Euler crossing the Königsberg bridges,

to Riemann counting prime numbers and to Gauss surveying the earth. There is poetry in this story, but the future is as important as the past. Many royal families die out, only a few survive.

My equation is about forty years old and since that time it has found fascinating and totally unexpected applications in fundamental physics, which Weyl would have both understood and appreciated. In fact many of the key ideas can be traced back to Weyl's own work.

Finally, on a more personal note, my equation embodies extensive collaboration with many of my colleagues: Fritz Hirzebruch from Bonn, Raoul Bott from Harvard, Is Singer from MIT and Vijay Patodi from Bombay, who like many talented poets died at a tragically early age. Beauty is a human experience and is best shared with friends.